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2009 J. Phys.: Condens. Matter 21 495306

(http://iopscience.iop.org/0953-8984/21/49/495306)

J. Phys.: Condens. Matter **21** (2009) 495306 (9pp) doi:10.1088/0953-8984/21/49/495306

Full counting statistics for noninteracting fermions: exact finite-temperature results and generalized long-time approximation

K Schönhammer

Physik-Department, Technische Universität München, D-85748 Garching, Germany
and
Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1,
D-37077 Göttingen, Germany

E-mail: schoenh@theorie.physik.uni-goettingen.de

Received 13 August 2009, in final form 30 September 2009
Published 19 November 2009
Online at stacks.iop.org/JPhysCM/21/495306

Abstract
Exact numerical results for the full counting statistics (FCS) of a one-dimensional tight-binding model of noninteracting electrons are presented at finite temperatures using an identity recently published by Abanov and Ivanov. A similar idea is used to derive an explicit expression for the cumulant generating function for a system consisting of two quasi-one-dimensional leads connected by a quantum dot in the long-time limit, generalizing the Levitov–Lesovik formula for two single-channel leads to systems with an arbitrary number of transverse channels.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The theory of noise in quantum transport in mesoscopic systems is a very active field of research [1, 2]. In addition to the first few moments of the transmitted charge the full probability distribution can be addressed, called *full counting statistics* (FCS). The systems usually studied consist of a finite 'dot' region connected to N leads which initially are separated from the dot region and have different chemical potentials [3–6]. After connecting the subsystems the time evolution of the particle transfer between the leads is studied.

In this paper we focus on systems with two quasi-one-dimensional noninteracting leads. The 'left' lead consists of M_{L} transverse channels and the initial state is described by a grand canonical ensemble with chemical potential μ_{L} and temperature $T_{\mathrm{L}} = \beta_{\mathrm{L}}/k_{\mathrm{B}}$. This lead is connected via a finite dot region to the 'right' lead with $M_{\mathrm{R}} = M - M_{\mathrm{L}}$ transverse channels, chemical potential $\mu_{\mathrm{R}} \leqslant \mu_{\mathrm{L}}$ and temperature $T_{\mathrm{R}} = \beta_{\mathrm{R}}/k_{\mathrm{B}}$.

For noninteracting electrons the calculation of the characteristic function of the probability function of the transferred charge can be exactly reduced to the evaluation of a time-dependent determinant over the full one-particle Hilbert space [7]. For the lattice systems studied in this paper the dimension of this space is finite before taking the thermodynamic limit. Exact results can be obtained for very large but finite systems by numerically calculating this determinant. For times t smaller than the time it takes the 'charge fronts' which move into the subsystems after connecting them to return to the connection point after reflection at the boundaries the results are almost independent of the size of the system [8].

The long-time limit was investigated by Levitov and Lesovik [3] for infinite leads. They presented the leading-time order result for the logarithm of the characteristic function (linear in t) as an energy integral over the logarithm of a determinant of an $M \times M$ matrix which involves the scattering matrix $s(\epsilon)$ for a single particle. For two single-channel leads, i.e. a strictly one-dimensional system, the 2×2 determinant was explicitly evaluated [3]. The result involving the transmission probability $T(\epsilon)$ and the Fermi functions of the two leads is usually called the 'Levitov–Lesovik formula'. In order to contrast it from the general leading long-time approximation involving the determinant of an $M \times M$ matrix it is called the 'two single-channel lead Levitov–Lesovik formula' in the following. For perfect transmission this formula leads at zero temperature to a delta function for the probability distribution of the transferred

charge incorrectly signalling 'zero shot noise'. It was shown previously that, in the case of perfect transmission at zero temperature, the logarithm of the characteristic function increases *logarithmically* with time, leading to a probability distribution of finite width also at zero temperature [8, 9].

In addition to exact numerical calculations for $M = 2$ an explicit result for the temperature-dependent Levitov–Lesovik $M \times M$ determinant is presented in terms of the eigenvalues of a temperature-independent matrix. The result is a sum of terms in the form appearing in the two single-channel lead Levitov–Lesovik formula and might have been guessed from the well-known results for the first two cumulants for this general case [1].

For the strictly one-dimensional case $M_{\mathrm{L}} = M_{\mathrm{R}} = 1$ exact numerical results were obtained for large but finite lattices at zero temperature [8]. The first step to obtain the probability distribution of the number of electrons transmitted to the right lead was to calculate the time dependence of the one-particle projection operator $P_{\mathrm{R}}(t)$ onto the right lead. At zero temperature only $\bar{P}_{\mathrm{R}}(t) = \bar{n}_0 P_{\mathrm{R}}(t)\bar{n}_0$ enters, where $\bar{n}_0$ is the projection operator onto the initially occupied one-particle states. The eigenvalues $p_m(t)$ of $\bar{P}_{\mathrm{R}}(t)$ determine the probability distribution w_{R} of the number of particles transferred to the right lead. The time-dependent entanglement entropy after connecting the subsystems can also be simply expressed in terms of these eigenvalues [10].

There are approximately $N_t = t(\mu_{\mathrm{L}} - \mu_{\mathrm{R}})/(2\pi)$ eigenvalues $p_m(t) \approx T(\mu_{\mathrm{R}} + 2\pi(m - 1/2)/t)$ different from zero and one in the long-time limit, where $T(\epsilon)$ is the transmission probability [8]. The transition region of the finite eigenvalues to the zero eigenvalues is not captured by this expression. In order to obtain analytical approximations for the exact numerical eigenvalues in this regime the logarithmic correction in the large time limit has to be known.

For finite temperatures the probability distribution is determined by $P_{\mathrm{R}}(t)$ in the full one-particle Hilbert space. A clever rewriting of the determinantal expression for the characteristic function [11] allows a simple generalization of the zero-temperature numerical procedure to obtain finite-temperature results. In section 2 exact results are presented for a strictly one-dimensional tight-binding model, i.e. $M_{\mathrm{L}} = M_{\mathrm{R}} = 1$. In the long-time limit an accurate analytical approximation for the eigenvalues of the temperature-dependent operator $X(t)$ introduced by Adamov and Ivanov (AI) [11] is presented which replaces $\bar{P}_{\mathrm{R}}(t)$ at finite temperatures.

The analytical expression for the eigenvalues of $X(t)$ is extended to arbitrary values of M_{L} and M_{R} in section 4 by using a similar rewriting as used by Adamov and Ivanov [11] for the leading order in t result for the logarithm of the characteristic function g_{R} corresponding to w_{R}. This derivation also provides a simple derivation of the explicit generalized Levitov–Lesovik formula already mentioned. For the special case $M_{\mathrm{R}} = 2$ a comparison is made with analytical results derived earlier [12].

In section 5 the generalized long-time approximation is elucidated for a model with leads which are stripes of equal width and, apart from a single site impurity, a perfect transition region.

2. Counting statistics for noninteracting electrons

2.1. General formulation

In the following we consider a system which consists of a finite 'dot' region described by the Hamiltonian H_0^{dot} connected to the left and right lead with Hamiltonians $H_{0,a}$ with $a = \mathrm{L}, \mathrm{R}$. The leads are initially separated from the dot region. The number of electrons in the initial state are N_0^{dot} and $N_{0,a}$. We assume the initial state $|\Phi(0)\rangle$ to be an eigenstate of H_0^{dot} and the $H_{0,a}$:

$$|\Phi(0)\rangle = |E_i^{N_0^{\mathrm{dot}}}\rangle \otimes |E_n^{N_{0,\mathrm{L}}}\rangle \otimes |E_p^{N_{0,\mathrm{R}}}\rangle. \tag{1}$$

The time evolution for times greater than zero is described by the Hamiltonian

$$H = H_0^{\mathrm{dot}} + H_{0,\mathrm{L}} + H_{0,\mathrm{R}} + \sum_a V_a \equiv H_0 + V. \tag{2}$$

The term V which couples the leads with the dot region will be specified later. The probability distribution that Q electrons are transferred to the right system after time t is given by

$$\begin{aligned} w_{\mathrm{R}}(t, Q) &= \langle\Phi(t)|\delta[Q - (\mathcal{N}_{\mathrm{R}} - N_{0,\mathrm{R}})]|\Phi(t)\rangle \\ &= \frac{1}{2\pi}\int \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda Q} g_{\mathrm{R}}(t, \lambda). \end{aligned} \tag{3}$$

Here $\mathcal{N}_{\mathrm{R}}$ is the particle number operator of the right lead and $g_{\mathrm{R}}(t, \lambda)$ is the characteristic function. With the particle number operators $\mathcal{N}_a(t)$ in the Heisenberg picture it is given by

$$g_{\mathrm{R}}(t, \lambda) = \langle\Phi(0)|\mathrm{e}^{\mathrm{i}\lambda\mathcal{N}_{\mathrm{R}}(t)}\mathrm{e}^{-\mathrm{i}\lambda\mathcal{N}_{\mathrm{R}}}|\Phi(0)\rangle. \tag{4}$$

The assumption that the initial state is an eigenstate of the particle number operators was used. For initially grand canonical subensembles with different temperatures and chemical potentials

$$\rho_0^{(a)} = \frac{\mathrm{e}^{-\beta_a(H_{0,a} - \mu_a\mathcal{N}_a)}}{\mathrm{Tr}_a\,\mathrm{e}^{-\beta_a(H_{0,a} - \mu_a\mathcal{N}_a)}}, \tag{5}$$

and ρ_0^{dot} of the same type, which corresponds to a total statistical operator ρ_0 of the generalized canonical form $\rho_0 = \mathrm{e}^{-\bar{H}_0}/\bar{Z}_0$ the averaging yields for the characteristic function

$$g_{\mathrm{R}}(t, \lambda) = \langle\mathrm{e}^{\mathrm{i}\lambda\mathcal{N}_{\mathrm{R}}(t)}\mathrm{e}^{-\mathrm{i}\lambda\mathcal{N}_R}\rangle, \tag{6}$$

where $\langle\cdots\rangle$ denotes the averaging with the statistical operator ρ_0. This result is also valid for interacting fermions.

2.2. Noninteracting fermions

For noninteracting fermions the characteristic function can be expressed as a determinant in the one-particle Hilbert space using Klich's trace formula [7, 8]:

$$\begin{aligned} g_{\mathrm{R}}(t, \lambda) &= \det[1 + (\mathrm{e}^{\mathrm{i}\lambda P_{\mathrm{R}}(t)}\mathrm{e}^{-\mathrm{i}\lambda P_{\mathrm{R}}} - 1)\bar{n}_0] \\ &\equiv \det[1 + a(t)\bar{n}_0] = \det[1 + \bar{n}_0 a(t)], \end{aligned} \tag{7}$$

where $\bar{n}_0 = (\mathrm{e}^{\bar{h}_0} + 1)^{-1}$ is the Fermi operator. It is determined by the Fermi functions describing the initial state. The equality in the second line holds because the inverse of $\bar{n}_0$ exists, in

contrast to the zero-temperature case where $\bar{n}_0$ is a projection operator.

Using $e^{i\lambda P_R(t)} = 1 + (e^{i\lambda} - 1)P_R(t)$ and the definition $d(\lambda) = e^{i\lambda} - 1$ the operator $1 + \bar{n}_0 a$ can be written in the form proposed by AI [11]:

$$\begin{aligned} 1 + \bar{n}_0 a &= [e^{i\lambda P_R} + \bar{n}_0(e^{i\lambda P_R(t)} - e^{i\lambda P_R})]e^{-i\lambda P_R} \\ &= \{1 + d(\lambda)[(1 - \bar{n}_0)P_R + \bar{n}_0 P_R(t)]\}\, e^{-i\lambda P_R} \\ &\equiv [1 + d(\lambda)X(t)]e^{-i\lambda P_R}. \end{aligned} \tag{8}$$

As $\bar{n}_0^{-1/2}$ exists and $\bar{n}_0$ and P_R commute it is more convenient to work with $\tilde{X}(t) = \bar{n}_0^{-1/2} X(t) \bar{n}_0^{1/2}$, i.e.

$$\tilde{X}(t) = (1 - \bar{n}_0)P_R + \bar{n}_0^{1/2} P_R(t) \bar{n}_0^{1/2}. \tag{9}$$

This yields for the characteristic function for arbitrarily large but finite systems

$$\begin{aligned} g_R(t, \lambda) &= e^{-i\lambda N_R} \det[1 + d(\lambda)\tilde{X}(t)] \\ &= e^{-i\lambda N_R} \prod_{m=1}^{N_H} [1 + (e^{i\lambda} - 1)\tilde{X}_m(t)] \\ &= \sum_{n=0}^{N_H} c_n(t) e^{i(n - N_R)\lambda}, \end{aligned} \tag{10}$$

where N_H is the dimension of the total one-particle Hilbert space and N_R of the one of the right lead. They are both finite for finite lattice systems. The coefficients $c_m(t)$ can be obtained recursively from the eigenvalues $\tilde{X}_n(t)$ of the AI one-particle operator $\tilde{X}(t)$ defined in equation (9) as described briefly in the appendix.

Apart from the replacements $N_{0,R} \to N_R$, $N_{tot} \to N_H$ and $p_m(t) \to \tilde{X}_m(t)$ this finite-temperature result has the same form as the $T = 0$ approach which was used as the starting point for the exact numerical calculation of the FCS [8]. The probability distribution at finite temperatures is given by

$$w_R(t, Q) = \sum_{n=1}^{N_H} c_n(t) \delta(Q - (n - N_R)). \tag{11}$$

To obtain exact results for the FCS one first has to calculate $\tilde{X}(t)$ using the result for $P_R(t)$ and then obtain its eigenvalues $\tilde{X}_m(t)$. This is done for the simplest case $M_L = M_R = 1$ in the following section.

3. Exact results for $M_L = M_R = 1$

3.1. The model

In this section we present exact numerical results for the probability distribution $w(t, Q)$ for a one-dimensional tight-binding model with a one-site (noninteracting) dot. The unperturbed one-particle Hamiltonians of the subsystems are given by

$$\begin{aligned} h_{0,a} &= -t_\parallel \sum_{m=1}^{N_a - 1} (|am\rangle\langle a(m+1)| + \text{h.c.}), \\ h_0^{dot} &= V_0 |0\rangle\langle 0|. \end{aligned} \tag{12}$$

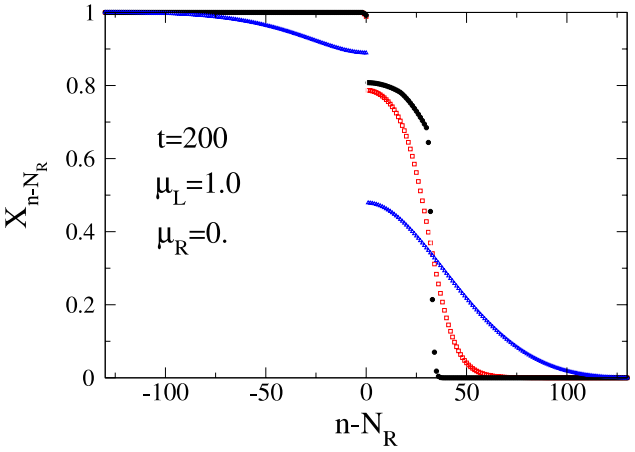

Figure 1. Eigenvalues of $\tilde{X}(t)$ which differ from one and zero for $t_L = 0.8$, $t_R = 0.5$ and $V_0 = 0.4$ at time $t = 200$ for three different temperatures, i.e. values of β. The full circles correspond to $\beta = 2000$, which hardly differs from the zero-temperature result. The open squares are the results for $\beta = 10$ and the open triangles for $\beta = 2$.

The number of sites in the leads are given by N_a. In $|am\rangle$ the label a takes the value 1 for $a = R$ and -1 for $a = L$. The hopping matrix elements in the leads $t_\parallel$ are taken as unity in the numerical calculations which leads to a total bandwidth of 4. The eigenstates of the unconnected leads are standing waves. The coupling between the subsystems is described by the hopping term

$$v = -t_L |-1\rangle\langle 0| - t_R |0\rangle\langle 1| + \text{h.c.} \tag{13}$$

3.2. Numerical results

The first step to calculate the eigenvalues $\tilde{X}_m(t)$ is to obtain $P_R(t)$ using the time dependence of the one-particle states $|\epsilon_\alpha^{(0)}\rangle$:

$$\langle \epsilon_\alpha^{(0)} | P_R(t) | \epsilon_\beta^{(0)} \rangle = \sum_{m=1}^{N_R} \langle \epsilon_\alpha^{(0)}(t) | m \rangle \langle m | \epsilon_\beta^{(0)}(t) \rangle. \tag{14}$$

The time dependence of the states is calculated using the spectral decomposition of the full one-particle Hamiltonian [8]. From the resulting $N_H \times N_H$ matrix one obtains $\tilde{X}(t)$ as prescribed in equation (9). In the following we show results for identical temperatures in the initial subsystems and $\mu_L = \mu_{dot} > \mu_R$.

We begin with a generic example $t_L = 0.8$, $t_R = 0.5$ and $V_0 = 0.4$. As the coupling of the dot to the leads is asymmetric the transmission probability at the resonance energy is less than one. The results shown are for time $t = 200$. If twice the number of lead sites N_a is larger than $v_{max} t = 2t$ the results for the eigenvalues $\tilde{X}_n$, which differ from zero and one, become independent of the N_a. For the exact numerical results shown we used $N_L = N_R = 500$. As $\tilde{X}(0) = P_R$ holds there are N_R eigenvalues of one and $N_L + 1$ eigenvalues of zero at the initial time. Therefore we show the eigenvalues in descending order as a function of $n - N_R$. In figure 1 the eigenvalues $\tilde{X}_n$ are shown for three different temperatures.

As mentioned in section 1 the 'zero-temperature' result $\beta = 2000$ can be well described analytically except for the narrow transition region to the zero eigenvalues which is related to the logarithmic corrections in the long-time limit [8, 9]. For $\beta = 10$ the main effect is to smooth out this transition region. A new effect sets in at larger temperatures. For $\beta = 2$ part of the eigenvalues with $n < N_R$ which are one at the initial time get visibly reduced.

In order to understand this behaviour analytically we start from the Levitov–Lesovik formula [3, 4]:

$$\ln g_R(t,\lambda) = \frac{t}{2\pi}\int_{-B}^{B} \ln(1+F(\epsilon,\lambda))\,d\epsilon, \qquad (15)$$

where $B = 2$ for the choice $t_\parallel = 1$ and

$$F(\epsilon,\lambda) = T(\epsilon)(d(\lambda) f_L(\epsilon)\bar{f}_R(\epsilon) + d^*(\lambda) f_R(\epsilon)\bar{f}_L(\epsilon)), \quad (16)$$

with $\bar{f}_a \equiv 1 - f_a$. We approximate the integral in equation (15) by a finite Riemann sum over N intervals of size $2B/N$ and use the trapezoidal rule

$$\ln g_R \approx \frac{t}{2\pi}\frac{2B}{N} \times \sum_{j=1}^{N} \ln\left[1 + F\left(-B + \left(j-\frac{1}{2}\right)\frac{2B}{N}, \lambda\right)\right], \qquad (17)$$

which agrees with the integral in the limit $N \to \infty$. As equation (15) is itself an approximation for the large-time limit we choose $N = N(t)$ with

$$N(t) = \frac{t}{2\pi} 2B. \qquad (18)$$

Then the prefactor in the sum equals one and $g_R(t)$ takes a form which can easily be compared with equation (10):

$$g_R(t,\lambda) \approx \prod_{j=1}^{N(t)} [1 + F(\epsilon_j,\lambda)], \qquad (19)$$

where $\epsilon_j \equiv -B + 2\pi(j-1/2)/t$.

In the low temperature regime $k_B T_a \ll \mu_L - \mu_R$ the approximation for the eigenvalues $\tilde{X}_j$ can easily be read off as the factor $f_R \bar{f}_L$ multiplying d^* in $F(\epsilon,\lambda)$ in equation (16) is exponentially small. With $e^{-i\lambda}(1 + d(\lambda)) = 1$ the comparison with equation (10) shows that the eigenvalues different from one are given by

$$\tilde{X}_j \approx T(\epsilon_j) f_L(\epsilon_j)\bar{f}_R(\epsilon_j) \approx T(\epsilon_j)(f_L(\epsilon_j) - f_R(\epsilon_j)). \quad (20)$$

At zero temperature this agrees with the result mentioned in section 1 [8].

At arbitrary temperatures one has to factor $1 + F(\epsilon,\lambda)$ with F defined in equation (16) in the form

$$1 + F(\epsilon,\lambda) = e^{-i\lambda}(1 + a_+(\epsilon)d(\lambda))(1 + a_-(\epsilon)d(\lambda)). \quad (21)$$

The comparison with equation (16) yields

$$a_\pm = \frac{1 + T(f_L - f_R)}{2} \pm w \qquad (22)$$

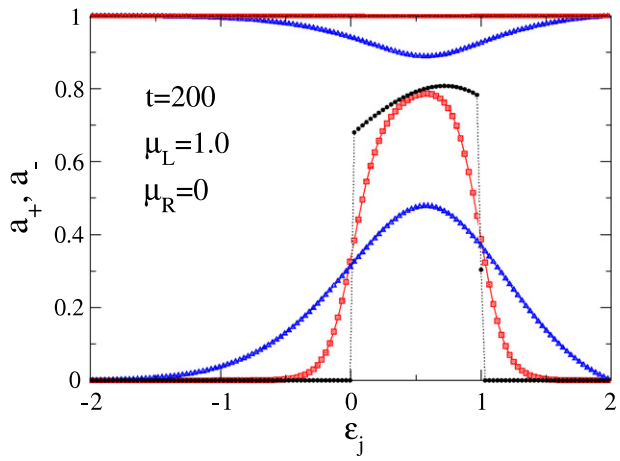

Figure 2. Long-time approximation equation (22) for the eigenvalues $\tilde{X}(t)$ plotted as a function of ϵ_j for the parameter values of figure 1. Filled dots: $\beta = 2000$, open squares: $\beta = 10$, open triangles: $\beta = 2$.

with

$$w = \sqrt{\left(\frac{1 + T(f_L - f_R)}{2}\right)^2 - T f_L \bar{f}_R} = \sqrt{\left(\frac{1 - T(f_L - f_R)}{2}\right)^2 - T f_R \bar{f}_L} = \sqrt{\left(\frac{1 - T(f_L + f_R)}{2}\right)^2 + T(1-T) f_L f_R}. \qquad (23)$$

The second form for w is useful for the discussion of the low temperature results and the third form shows that there is an energy gap in the spectrum for non-perfect transmission. It also shows that the factorization is simplest for perfect transmission:

$$1 + F(\epsilon,\lambda) = e^{-i\lambda}(1 + f_L(\epsilon)d(\lambda))(1 + \bar{f}_R(\epsilon)d(\lambda)). \quad (24)$$

In figure 2 we show the analytical approximation equation (22) for the eigenvalues for the parameter values used in figure 1 as a function of the ϵ_j. The result for $\beta = 2000$ (filled dots) is almost identical to the zero-temperature results. The eigenvalues $a_-(\epsilon_i)$ are nonzero in the energy range between μ_R and μ_L and show the energy dependence of the transmission probability. The eigenvalues $a_+(\epsilon_i)$ are almost identical to one. Also for $\beta = 10$ (open squares) the $a_+(\epsilon_i)$ equals one within the drawing accuracy while the $a_-(\epsilon_i)$ are very well approximated by equation (20) and show the thermal broadening of the zero-temperature result. For $\beta = 2$ (open triangles) equation (20) no longer presents a good approximation for the $a_-(\epsilon_i)$ and the $a_+(\epsilon_i)$ are clearly smaller than one in a large energy range.

In order to compare the approximate eigenvalues $a_\pm$ with the exact numerical eigenvalues $\tilde{X}_n$ the $a_\pm$ have to be brought into descending order. This is shown in figure 3. For $\beta = 2$ the a_- (open triangles) agree very well with the exact results apart from the fact that there are pairs of almost equal eigenvalues. As shown later this has a rather small effect for the calculation

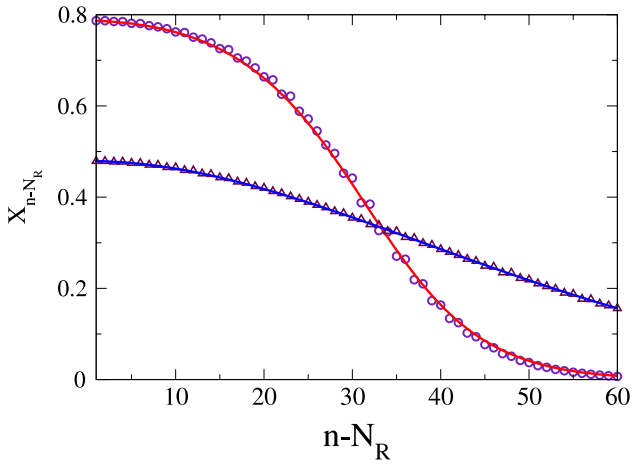

Figure 3. Comparison of the exact eigenvalues of figure 1 (shown as full lines) with the approximations a_- presented in descending order. The results for $\beta = 10$ and 2 are shown in a restricted range.

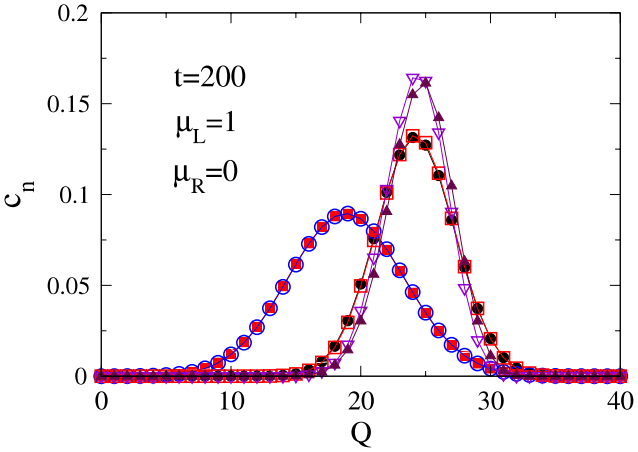

Figure 4. Weights of the probability distribution $w_R(t, Q)$ for the parameters of figure 1. The exact results are presented by the full symbols, the approximation using equation (22) by the open ones: $\beta = 2$: filled squares, open circles, $\beta = 10$: filled circles, open squares, $\beta = 2000$: filled triangles, open triangles.

of the probability distribution w_R. For $\beta = 10$ the deviations are a bit larger. As shown earlier the transition region to the zero eigenvalues for $\beta = 2000$ is not captured by the approximation $T(\epsilon_i)$ [8].

In order to obtain the probability distribution $w_R(t, Q)$ from the eigenvalues $\tilde{X}_n$ the product in equation (10) has to be evaluated recursively. The details are presented in the appendix. The results for the parameters of figure 1 are shown in figure 4. For $\beta = 2$ the exact result (full squares) agrees very well with the one using the approximation of equation (22) (open squares). For $\beta = 10$ the agreement of the exact result (full circles) with the approximation (open squares) is still rather good. For $\beta = 2000$ the filled and open symbols no longer overlap.

The discrepancy between the exact results and the approximation equation (22) is most prominent for perfect transmission in the zero-temperature limit. While the Levitov–Lesovik formula predicts a single delta function of weight one

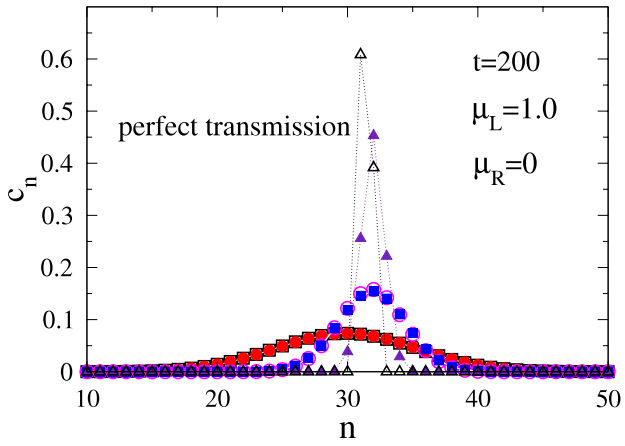

Figure 5. Weights of the probability distribution $w_R(t, Q)$ for the case of perfect transmission. The exact results are presented by the full symbols, the approximation using equation (22) by the open ones: $\beta = 2$. filled circles, open squares, $\beta = 10$: filled squares, open circles, $\beta = 2000$: filled triangles, open triangles.

('zero shot noise') the exact result clearly has a finite width [8] as shown in figure 5, where the $t_L = 1$, $t_R = 1$ and $V_0 = 0$ was used leading to perfect transmission.

3.3. Perfect transmission

As the discrepancy for very low temperatures concerns the width of the approximate distribution shown in figure 5 it is useful to discuss the behaviour of the second order cumulant:

$$\begin{aligned}\kappa_2(t) &= \sum_{n=1}^{N_H} \tilde{X}_n(t)(1-\tilde{X}_n(t)) \\ &\approx \sum_{j=1}^{N(t)} [a_+(\epsilon_j)(1-a_+(\epsilon_j)) + a_-(\epsilon_j)(1-a_-(\epsilon_j))] \\ &= \sum_{j=1}^{N(t)} [T^2(f_L\bar{f}_L + f_R\bar{f}_R) + T(1-T)(f_L\bar{f}_R + f_R\bar{f}_L)].\end{aligned} \tag{25}$$

The (leading time order) shot noise contribution [13] proportional to $T(1-T)$ vanishes for perfect transmission and the remaining term is well known in the limit where the sum is replaced again by the integral [1, 2, 4].

Because of the factorization in equation (24) of $1 + F$ for perfect transmission the derivative of $\ln g_R$ with respect to $i\lambda$ takes the simple form

$$\begin{aligned}\frac{2\pi}{t}\frac{\partial \ln g_R(t,\lambda)}{\partial(i\lambda)} &= \mu_L - \mu_R + \left(\frac{1}{\beta_R} + \frac{1}{\beta_L}\right) i\lambda \\ &+ \sum_a \frac{1}{\beta_a} \ln\left[\frac{1+e^{-\beta_a(B-a\mu_a)-i\lambda}}{1+e^{-\beta_a(B+a\mu_a)+i\lambda}}\right],\end{aligned} \tag{26}$$

where the factor a in $a\mu_a$ again takes the value 1 for $a = R$ and -1 for $a = L$. In the wide band limit $\beta_a(B - |\mu_a|) \gg 1$ the logarithmic terms can be neglected and w_R is a Gaussian in the Levitov–Lesovik approximation with a temperature-independent mean value [4]. The fact that the average charge transfer for $\beta = 2$ in figure 5 is slightly less than for $\beta = 10$

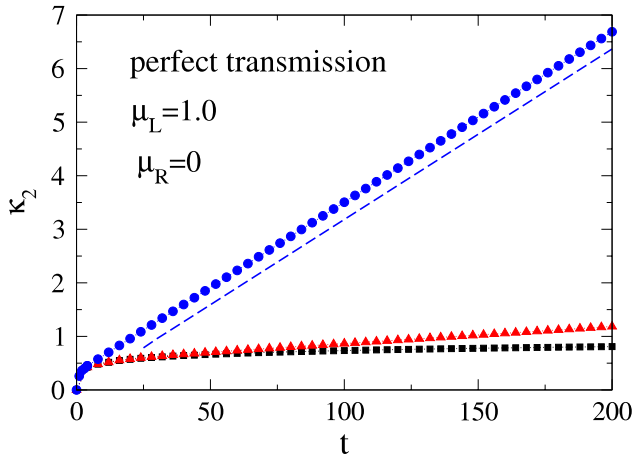

Figure 6. Exact results (filled symbols) for second-order cumulant κ_2 for perfect transmission as a function of time for three different temperatures: $\beta = 2000$: filled squares, $\beta = 100$: filled triangles and $\beta = 10$: filled circles. The dotted line shows the linear part $t/(\pi\beta)$ for $\beta = 10$.

and $\beta = 2000$ is related to the correction term in equation (26) which adds a constant contribution for $\lambda = 0$.

In the wide band limit $\ln g_{\rm R}(t, \lambda)$ can be calculated analytically also for an energy-independent transmission probability which differs from one [14].

Exact numerical results for κ_2 are shown in figure 6 as a function of time for system size $N_{\rm L} = N_{\rm R} = 500$ up to times where the result is independent of this system size. For $\beta = 10$ (filled circles) there is an almost linear increase of κ_2 rather quickly. The dotted line shows the linear increase which follows from the finite-temperature Levitov–Lesovik formula (see equation (26)). For $\beta = 100$ (filled triangles) there is crossover from a logarithmic increase to a linear time dependence around $t \approx 50$. For $\beta = 2000$ the times shown are too small to see the corresponding crossover and the shown increase is logarithmic in time. This logarithmic behaviour [8, 9, 15] is not captured by the approximate eigenvalues $a_\pm$ in equation (22).

4. Long-time behaviour for general values of $M_{\rm L}$ and $M_{\rm R}$

To study the long-time limit one first takes the thermodynamic limit in which the energies of the unperturbed eigenstates $|\epsilon, a, i\rangle$ with $i = 1, \ldots, M_a$ of the left and right lead form $M_{\rm L}$ and $M_{\rm R}$ continua extending from $\epsilon^{a,i}_{\rm min}$ to $\epsilon^{a,i}_{\rm max}$ which usually partially overlap. For a given energy ϵ the 'open scattering channels' [16] in the leads are those for which $\epsilon \in [\epsilon^{a,i}_{\rm min}, \epsilon^{a,i}_{\rm max}]$. Their number is denoted as $M_a(\epsilon) \leqslant M_a$. For a given ϵ we order the channels with $i = 1, \ldots, M_a(\epsilon)$ as the open channels and $i = M_a(\epsilon) + 1, \ldots, M_a$ as the closed channels. The dimension of the scattering matrix $s(\epsilon)$ for a single particle [16] is given by $M(\epsilon) = M_{\rm L}(\epsilon) + M_{\rm R}(\epsilon) \leqslant M$, with $M_a(\epsilon) \geqslant 1$ in order to have scattering at all.

In the following we use a Dirac notation in the $M(\epsilon)$-dimensional space with the orthonormal basis $|a, i)$ where the i run from 1 to $M_a(\epsilon)$. Then the projection operators $\tilde{P}_a(\epsilon)$ on the lead channels are

$$\tilde{P}_a(\epsilon) = \sum_{i=1}^{M_a(\epsilon)} |a, i)(a, i|. \tag{27}$$

The energy dependence of the $\tilde{P}_a(\epsilon)$ is only via selection of the open channels.

For the general geometry discussed in section 2 the long-time linear in t contribution to $\ln g_{\rm R}$ takes the form [3, 12]

$$\ln g_{\rm R}(t, \lambda) = \frac{t}{2\pi} \int_{\epsilon_{\rm min}}^{\epsilon_{\rm max}} \ln \det[1 + c(\epsilon, \lambda)]\, {\rm d}\epsilon. \tag{28}$$

Here $c(\epsilon, \lambda)$ is the $M(\epsilon) \times M(\epsilon)$ matrix:

$$c(\epsilon, \lambda) = [s^\dagger(\epsilon) e(\lambda, \epsilon) s(\epsilon) {\rm e}^\dagger(\lambda, \epsilon) - 1] f(\epsilon), \tag{29}$$

with

$$\begin{aligned} e(\lambda, \epsilon) &= \tilde{P}_{\rm L}(\epsilon) + {\rm e}^{{\rm i}\lambda} \tilde{P}_{\rm R}(\epsilon) = 1 + d(\lambda) \tilde{P}_{\rm R}(\epsilon) \\ f(\epsilon) &= f_{\rm L}(\epsilon) \tilde{P}_{\rm L}(\epsilon) + f_{\rm R}(\epsilon) \tilde{P}_{\rm R}(\epsilon). \end{aligned} \tag{30}$$

The values of $\epsilon_{\rm min}$ and $\epsilon_{\rm max}$ depend on the details of the microscopic model.

Now one can essentially repeat the steps used to derive the AI-form equation (10) to obtain

$$\det[1 + c(\epsilon, \lambda)] = {\rm e}^{-{\rm i}\lambda M_{\rm R}(\epsilon)} \det[1 + d(\lambda) \tilde{X}(\epsilon)] \tag{31}$$

with

$$\tilde{X} = (1 - f) \tilde{P}_{\rm R} + f^{1/2} s^\dagger \tilde{P}_{\rm R} s f^{1/2}. \tag{32}$$

The determinant in equation (31) can be calculated by first solving the eigenvalue problem for $\tilde{X}(\epsilon)$. As the $M(\epsilon) \times M(\epsilon)$ matrix $\tilde{X}(\epsilon)$ in equation (32) is temperature-dependent it looks as if one has to solve a different eigenvalue problem for each temperature. In the following we show that this is *not* necessary. In fact, it is sufficient to solve a *single* $M_{\rm R}(\epsilon) \times M_{\rm R}(\epsilon)$ eigenvalue problem to obtain a new generalized long-time approximation for arbitrary temperatures. To show this we write

$$\tilde{X}(\epsilon) = \sum_{j=1}^{M_{\rm R}(\epsilon)} [\bar{f}_{\rm R} |R, j)(R, j| + |b_j)(b_j|] \tag{33}$$

with

$$|b_j) = f^{1/2} s^\dagger |R, j), \qquad (b_j| = (R, j| s f^{1/2}. \tag{34}$$

Multiplying the eigenvalue problem $\tilde{X}|X_\alpha) = X_\alpha |X_\alpha)$ from the left with $(b_i|$ yields

$$\sum_{j=1}^{M_{\rm R}(\epsilon)} [(b_i|R, j)(R, j|X_\alpha) \bar{f}_{\rm R} + (b_i|b_j)(b_j|X_\alpha)] = X_\alpha (b_i|X_\alpha) \tag{35}$$

and multiplying with $(R, i|$ gives for the overlaps $(R, i|X_\alpha)$:

$$(R, i|X_\alpha) = \frac{1}{X_\alpha - \bar{f}_{\rm R}} \sum_{j=1}^{M_{\rm R}(\epsilon)} (R, i|b_j)(b_j|X_\alpha). \tag{36}$$

Inserting this into equation (35) leads after multiplication with $X_\alpha - \bar{f}_R$ to

$$\sum_{j=1}^{M_R(\epsilon)} [\bar{f}_R(b_i|\tilde{P}_R|b_j) + (X_\alpha - \bar{f}_R)(b_i|b_j)](b_j|X_\alpha) = (X_\alpha^2 - \bar{f}_R X_\alpha)(b_i|X_\alpha). \quad (37)$$

This equation can be rewritten using

$$(b_i|b_j) = f_R\delta_{ij} + (f_L - f_R)(R, i|s\tilde{P}_L s^\dagger|R, j). \quad (38)$$

Therefore a single Hermitian operator in the $M_R(\epsilon)$-dimensional subspace spanned by the $|i\rangle \equiv |R, i)$ determines the original eigenvalue problem

$$\left[(f_L - f_R)X_\alpha - f_L\bar{f}_R\right]\tau|Y_\alpha^{(R)}\rangle = (X_\alpha^2 - X_\alpha)|Y_\alpha^{(R)}\rangle, \quad (39)$$

where $|Y_\alpha^{(R)}\rangle \equiv \tilde{P}_R s f^{1/2}|X_\alpha)$ and

$$\tau = \tilde{P}_R s\tilde{P}_L s^\dagger \tilde{P}_R = (\tilde{P}_R s\tilde{P}_L)(\tilde{P}_R s\tilde{P}_L)^\dagger \equiv AA^\dagger. \quad (40)$$

The elements of the matrix A are the left to right transmission amplitudes[1].

The matrix elements of τ are given by $(i, j \in [1, M_R(\epsilon)])$

$$\tau_{ij}(\epsilon) = \sum_{l=M_R(\epsilon)+1}^{M(\epsilon)} s_{il}(\epsilon)s_{lj}^\dagger(\epsilon). \quad (41)$$

After solving the eigenvalue problem $\tau|\tau_\mu\rangle = \tau_\mu|\tau_\mu\rangle$ the determination of the X_α in equation (39), after multiplying with $\langle\tau_\mu|$, is reduced to solving a quadratic equation. With $\alpha \to \mu, \pm$ the solution is

$$X_{\mu,\pm} = \frac{1 + \tau_\mu(f_L - f_R)}{2} \pm \sqrt{\left(\frac{1 + \tau_\mu(f_L - f_R)}{2}\right)^2 - \tau_\mu f_L\bar{f}_R}. \quad (42)$$

This is like equations (22) and (23) with $T(\epsilon_i)$ replaced by $\tau_\mu(\epsilon_i)$.

For $M_R(\epsilon) = 1$ there exists only one eigenvalue τ_1, which for $M_L(\epsilon) = 1$ is given by $|s_{12}(\epsilon)|^2 = T(\epsilon)$. For $M_R(\epsilon) = 1$ and arbitrary values of $M_L(\epsilon)$ the single eigenvalue is given by

$$\tau_1(\epsilon) = \sum_{l=2}^{M_{(\epsilon)}} |s_{1l}(\epsilon)|^2. \quad (43)$$

This corresponds to the simplest generalization of the Levitov–Lesovik formula [8].

For $M_R(\epsilon) = 2$ the two eigenvalues of $\tau(\epsilon)$ are given by

$$\tau_{1,2} = \frac{\mathrm{tr}\tau}{2} \pm \sqrt{\left(\frac{\mathrm{tr}\tau}{2}\right)^2 + \det\tau}. \quad (44)$$

Zero-temperature results for this case were presented earlier [12] using a different derivation. For the special case $M_L(\epsilon) = 1$ the determinant of τ vanishes and only one eigenvalue of τ is different from zero. Generally the number of eigenvalues of τ which differ from zero is less than or equal to $M_<(\epsilon)$, where $M_<(\epsilon)$ is the smaller of the two $M_a(\epsilon)$. This stems from the fact that $AA^\dagger$ and $A^\dagger A$ have the same nonvanishing eigenvalues with the same multiplicities. For $M_L(\epsilon) < M_R(\epsilon)$ one better calculates the eigenvalues of $A^\dagger A$ which is a Hermitian $M_L(\epsilon) \times M_L(\epsilon)$ matrix.

In order to compare our general result with the Levitov–Lesovik formula one can use

$$\begin{aligned} &e^{-i\lambda}(1 + d(\lambda)X_{\mu,+})(1 + d(\lambda)X_{\mu,-}) \\ &= e^{-i\lambda}[1 + (1 + (f_L - f_R)\tau_\mu)d(\lambda) + \tau_\mu\bar{f}_R f_L d(\lambda)^2] \\ &= 1 + \tau_\mu(d(\lambda)f_L\bar{f}_R + d(\lambda)^* f_R\bar{f}_L). \end{aligned} \quad (45)$$

With $\det(1 + d(\lambda)\tilde{X}_\alpha(\epsilon)) = \prod_\alpha(1 + d(\lambda)X_\alpha(\epsilon))$ this yields

$$\ln g_R \approx \frac{t}{2\pi} \times \int_{\epsilon_{\min}}^{\epsilon_{\max}} \sum_{\mu=1}^{M_<(\epsilon)} \ln[1 + \tau_\mu(df_L\bar{f}_R + d^* f_R\bar{f}_L)]\,d\epsilon. \quad (46)$$

This completes the derivation of the generalization of the Levitov–Lesovik formula for two general quasi-one-dimensional leads[2]. The integration has to be split up into N_I energy intervals from $\epsilon^{(i)}$ to $\epsilon^{(i+1)}$, where $i = 1, \ldots, N_I$ and $\epsilon^{(1)} = \epsilon_{\min}$ and $\epsilon^{(N_I+1)} = \epsilon_{\max}$. In the intervals the number of open channels determined by the $M_a(\epsilon)$ is constant. We denote the constant value of $M_<(\epsilon)$ in the mth interval by $M_<^{(m)}$. The splitting is discussed in the following section for a simple model system.

With the eigenvalues $X_\alpha(\epsilon)$ of the $M(\epsilon) \times M(\epsilon)$ matrix $\tilde{X}(\epsilon)$ and the transition from the integral to a finite Riemann sum as in equation (17) one can obtain an approximation for the eigenvalues of the operator $\tilde{X}(t)$ defined in equation (9) which generalizes the introduction of the $a_\pm$ in section 3. The approximation for g_R is

$$g_R(t, \lambda) \approx \prod_{I_m} e^{-i\lambda 2M_<^{(m)}N_m(t)} \prod_{j=1}^{N_m(t)} \prod_{\alpha=1}^{2M_<^{(m)}} [1 + d(\lambda)X_\alpha(\epsilon_j^{(m)})] \quad (47)$$

with $N_m(t) = (\epsilon^{(m+1)} - \epsilon^{(m)})t/2\pi$ and the energy variables are given by $\epsilon_j^{(m)} = \epsilon^{(m)} + 2\pi(j - 1/2)/t$. Therefore the approximation for the eigenvalues $\tilde{X}_m(t)$ which differ from 1 and 0 are given by $X_\alpha(\epsilon_j^{(m)})$. The time dependence enters via $\epsilon_j^{(m)}$.

5. Almost perfect stripe

In order to elucidate our general result equation (46) we consider leads which are both stripes of width $N_\perp$. Analytical results for the eigenvalues τ_μ are presented for the case where the dot region is identical to the leads except for a single site impurity:

$$h = -\sum_{m=-\infty}^{\infty}\sum_{n=1}^{N_\perp - 1}[t_\parallel|m, n\rangle\langle m + 1, n| + t_\perp|m, n\rangle\langle m, n + 1| + \text{h.c}] + V_0|0, n_0\rangle\langle 0, n_0| \equiv \tilde{h}_0 + \tilde{v}_0. \quad (48)$$

[1] The matrix A is usually denoted as t [1]. We chose not to call it t as we use this letter for the t operator.

[2] After submitting this paper the author was informed by Yu Nazarov of the appearance of the book [17]. There the leading-time order expression equation (46) is presented without a derivation and without mentioning the subtleties discussed here.

The tildes are introduced to indicate that the separation of the Hamiltonian in the unperturbed part and the perturbation is different from the one used in equations (2), (12) and (13).

For $V_0 = 0$ this is an ideal infinite stripe with eigenvalues

$$\epsilon_{k,l} = -2t_\parallel \cos k - 2t_\perp \cos\frac{l\pi}{N_\perp + 1} \equiv \epsilon_k^\parallel + \epsilon_l^\perp, \tag{49}$$

with $k \in [-\pi, \pi]$ and $l = 1, \ldots, N_\perp$. For a given energy ϵ there are $2N_\perp(\epsilon) \leqslant 2N_\perp$ scattering channels open for which $|\epsilon - \epsilon_l| \leqslant 2|t_\parallel|$ holds. The standing wave lead states $|\epsilon, a, l\rangle$ are labelled by $a = \mathrm{L}, \mathrm{R}$ and the transverse quantum numbers l of the open channels. The scattering matrix is obtained via the t operator [16]

$$t(z) = v + vg(z)v, \tag{50}$$

where $g(z) = (z - h)^{-1}$ is the exact resolvent and v is the generalization of the operator connecting the leads with the dot in equation (13). For the almost perfect stripe it is given by

$$\begin{aligned} v &= -t_\parallel \sum_{n=1}^{N_\perp} (|-1, n\rangle\langle 0, n| + |0, n\rangle\langle 1, n| + \mathrm{h.c}) \\ &= -t_\parallel \sum_{l=1}^{N_\perp} (|l_{(-1)}\rangle\langle l_{(0)}| + |l_{(0)}\rangle\langle l_{(1)}| + \mathrm{h.c}), \end{aligned} \tag{51}$$

where the $|l_{(m)}\rangle$ are the standing wave eigenstates in the perpendicular direction formed from the states $|m, n\rangle$. Then the t-matrix elements take the simple form

$$\begin{aligned} \langle \epsilon, a, l|t(z)|\epsilon, a', l'\rangle &= t_\parallel^2 \langle \epsilon, a, l|l_{(a)}\rangle \\ &\times \langle l_{(0)}|g(z)|l'_{(0)}\rangle\langle l'_{(a')}|\epsilon, a', l'\rangle, \end{aligned} \tag{52}$$

where $|l_{(a)}\rangle$ is the standing wave state at $n = -1$ for $a = \mathrm{L}$ and $n = 1$ for $a = \mathrm{R}$. The t-matrix elements enter the scattering matrix for $z = \epsilon + \mathrm{i}0$.

The exact resolvent matrix elements can easily be calculated for the Hamiltonian in equation (48) as the site impurity provides a separable perturbation. With $\tilde{g}_0(z) = (z - \tilde{h}_0)^{-1}$ one obtains

$$\begin{aligned} \langle l_{(0)}|g|l'_{(0)}\rangle &= \langle l_{(0)}|\tilde{g}_0|l'_{(0)}\rangle \\ &+ \frac{\langle l_{(0)}|\tilde{g}_0|0, n_0\rangle V_0 \langle 0, n_0|\tilde{g}_0|l'_{(0)}\rangle}{1 - V_0\langle 0, n_0|\tilde{g}_0|0, n_0\rangle}. \end{aligned} \tag{53}$$

For the open channels

$$\langle l_{(0)}|\tilde{g}_0(\epsilon + \mathrm{i}0)|l'_{(0)}\rangle = \delta_{ll'} \frac{-\mathrm{i}}{\sqrt{B_\parallel^2 - (\epsilon - \epsilon_l)^2}} \tag{54}$$

holds, with $B_\parallel = 2t_\parallel$. The $\tilde{g}_0$-matrix element in the denominator in equation (53) involves contributions from the open and closed channels:

$$\begin{aligned} \langle 0, n_0|\tilde{g}_0(\epsilon + \mathrm{i}0)|0, n_0\rangle &= \sum_{l(open)} \frac{-\mathrm{i}|\langle l_{(0)}|0, n_0\rangle|^2}{\sqrt{B_\parallel^2 - (\epsilon - \epsilon_l)^2}} \\ &+ \sum_{l(\mathrm{closed})} \frac{|\langle l_{(0)}|0, n_0\rangle|^2}{\sqrt{(\epsilon - \epsilon_l)^2 - B_\parallel^2}} \\ &\equiv -\mathrm{i}\pi\tilde{\rho}_{00} + \tilde{g}_{00}^R. \end{aligned} \tag{55}$$

While the contribution of the open channels is purely imaginary the one of the closed channels is real. To complete the calculation of the t-matrix elements in equation (52) the overlaps $\langle \epsilon, a, l|l_{(a)}\rangle$ are needed. They are related to the density of states at the boundary of a semi-infinite chain:

$$t_\parallel \langle \epsilon, a, l|l_{(a)}\rangle = \left(\frac{1}{2\pi}\sqrt{B_\parallel^2 - (\epsilon - \epsilon_l)^2}\right)^{1/2}. \tag{56}$$

This leads to the scattering matrix

$$s_{al,a'l'}(\epsilon) = \delta_{ll'}(\delta_{aa'} - 1) + (l|n_0)u(n_0|l') \tag{57}$$

with

$$\begin{aligned} (l|n_0) &= \frac{\langle l_{(0)}|0, n_0\rangle}{\sqrt{B_\parallel^2 - (\epsilon - \epsilon_l)^2}}, \\ u &= \frac{\mathrm{i}V_0}{1 - V_0\tilde{g}_{00}^R + \mathrm{i}\pi V_0\tilde{\rho}_{00}}. \end{aligned} \tag{58}$$

With the $N_\perp(\epsilon) \times N_\perp(\epsilon)$ projected scattering matrix $s_{\mathrm{RL}} = -1 + |n_0)u(n_0|$ the operator τ defined in equation (40) is given by

$$\tau = 1 - (u + u^* - (n_0|n_0)|u|^2)|n_0)(n_0|. \tag{59}$$

Because of the separable form of $\tau - 1$ the only eigenvalue of τ different from 1 is given by

$$\tau_1 = 1 - (u + u^* - (n_0|n_0)|u|^2)(n_0|n_0). \tag{60}$$

Using $(n_0|n_0) = \sum_{l(\mathrm{open})}(n_0|l)(l|n_0) = \tilde{\rho}_{00}/2$ one finally obtains

$$\begin{aligned} \tau_1 &= \frac{(1 - V_0\tilde{g}_{00}^R)^2}{(1 - V_0\tilde{g}_{00}^R)^2 + (\pi V_0\tilde{\rho}_{00})^2}, \\ \tau_2 &= \tau_3 = \cdots = \tau_{N_\perp(\epsilon)} = 1. \end{aligned} \tag{61}$$

The 'perfect transmission' eigenvalues $\tau_i = 1$ yield contributions to $\ln g_{\mathrm{R}}$ of the form discussed following equation (26). As the energy integrations in equation (46) for fixed $M_<(\epsilon)$ are over restricted energy ranges the logarithmic corrections in equation (26) are important.

The simplest case is a $N_\perp = 2$ ladder system. For the special case $t_\parallel = t_\perp = 1$ there are two bands corresponding to $\epsilon_l^\perp = \pm 1$ of width 4. Therefore one has to distinguish the energy intervals $[-3, -1]$, $[-1, 1]$ and $[1, 3]$ with one, two and one open channel. When both channels are open $\tau_2 = 1$ and $\tilde{g}_{00}^R$ vanishes. This implies $\tau_1 \to 0$ for $V_0 \to \infty$, while τ_1 stays finite in this limit when only one channel is open.

6. Summary

In this paper we have generalized the exact numerical method to calculate the FCS for large but finite systems [8] to finite temperatures using the eigenvalues of the operator $X(t)$ in the Hilbert space of a single particle introduced by Abanov and Ivanov [11]. In the long-time limit the results for the probability distribution for the number of transmitted particles agree well with the result using the Levitov–Lesovik approximation [3, 4] except for the case of (almost) perfect transmission.

Using a similar identity for the finite-temperature leading order in time result for the logarithm of the characteristic function a new explicit result for $\ln g_{\mathrm{R}}$ was presented in equation (46) for two general quasi-one-dimensional leads which involves the eigenvalues of a matrix formed from the transmission amplitudes. For a simple model these eigenvalues were calculated analytically.

Acknowledgment

The author would like to thank W Zwerger for the hospitality during his sabbatical stay at TU München.

Appendix. Recursive step in the calculation of $w_{\mathrm{R}}(t, Q)$

The numerical finite-temperature results presented in section 3 were obtained by first calculating the eigenvalues $\tilde{X}_m(t)$ and then performing the product in equation (10). This is done iteratively as follows.

Let $F_N(x)$ be a polynomial given in the form of a product

$$F_N(x) = \prod_{i=1}^{N}(a_i + b_i x) = \sum_{m=0}^{N} c_m^{(N)} x^m. \tag{A.1}$$

The coefficients $c_m^{(N)}$ are obtained iteratively by calculating the polynomials $F_M(x)$ with coefficients $c_m^{(M)}$ starting with $M = 1$ and using

$$\begin{aligned} F_{M+1}(x) &= (a_{M+1} + b_{M+1}x)F_M(x) \\ &= \sum_{m=0}^{M}(a_{M+1}c_m^{(M)}x^m + b_{M+1}c_m^{(M)}x^{m+1}). \end{aligned} \tag{A.2}$$

This leads to the recurrence relations

$$\begin{aligned} c_0^{(M+1)} &= a_{M+1}c_0^{(M)} \\ c_m^{(M+1)} &= a_{M+1}c_m^{(M)} + b_{M+1}c_{m-1}^{(M)}, \qquad 1 \leqslant m \leqslant M \\ c_{M+1}^{(M+1)} &= b_{M+1}c_M^{(M)}. \end{aligned} \tag{A.3}$$

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